Summary

-Mathematics in Economics and Business-

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Lecture 1

Functions of one variable.

Introduction to functions

Functions are used to simply describe relationships between multiple (economic) variables.

In the function $y = f(x)$ the variable x can be filled into the function, i.e. x is the input variable. The function will then give exactly one result for the variable y, i.e. the output variable.
The input variables are called **independent variables**, the output variables dependent on the input variables, and are called **dependent variables**.

The collection of all possible values of the independent variable (x) is the **domain** of a function. In addition, the set of all possible values of the function, i.e. the dependent variable (y), is called the **range** of a function.
The domain and range of a function in economics are usually intervals of real numbers. An interval can be notated as e.g. $(0,10]$, which is an interval of all real numbers between 0 and 10; <u>excluding</u> number 0, but <u>including</u> number 10.

In short, the function f is a rule which assigns to every point in domain of the function x, exactly one point in the range of the function, y.

Function properties

Monotonicity:
In what direction does the function change? Does it only increase or decrease?

The definition of monotonicity is the following:
$\forall\, x_1, x_2 \;\; s.t. \;\; x_2 > x_1$

- Function $f(x)$ is <u>strictly</u> **increasing** $\Leftrightarrow f(x_2) > f(x_1)$
- Function $f(x)$ is <u>strictly</u> **decreasing** $\Leftrightarrow f(x_2) < f(x_1)$

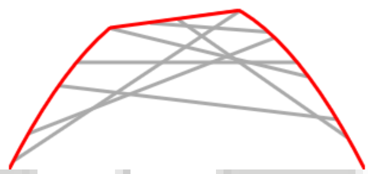
- Function $f(x)$ is <u>weakly</u> **increasing** $\Leftrightarrow f(x_2) \geq f(x_1)$
- Function $f(x)$ is <u>weakly</u> **decreasing** $\Leftrightarrow f(x_2) \leq f(x_1)$

- Function $f(x)$ is **constant** $\Leftrightarrow f(x_2) = f(x_1)$

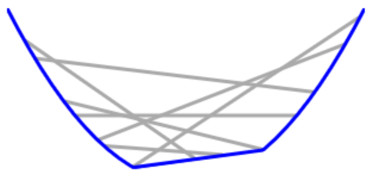
Curvature:
At what speed does the function increase or decrease?

Continuity:
Does the function $f(x)$ have some "holes" in it and/or jumps in its value? Or can we draw the function without lifting a pan from the paper?

Concave and convex
A function is <u>concave</u> if every line segment joining two points on its graph does not lie above the graph at any point. The function lies **ABOVE** the line segments. When the second order derivative (Lecture 2 – higher order derivatives) is smaller than 0, on a given interval, the function is considered concave.



A function is <u>convex</u> if every line segment joining two points on its graph does not lie below the graph at any point. The function lies **BELOW** the line segments. When the second order derivative (Lecture 2 – higher order derivatives) is greater than 0, on a given interval, the function is considered concave.

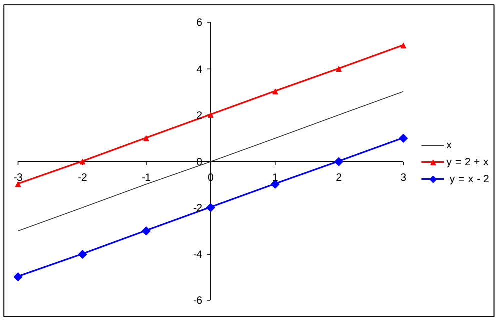


Inverse functions

It is possible to exchange the variables in functions, in order to find the inverse function. The inverse function f^{-1} of the function f is a function for which holds:
$x = f^{-1}(y) \Leftrightarrow y = f(x)$

Inverse functions are functions. Therefore, it can be said that for each x in the function's domain there must be exactly one y in the co-domain.

For example, find the inverse of the function $f(x) = 2 + x$



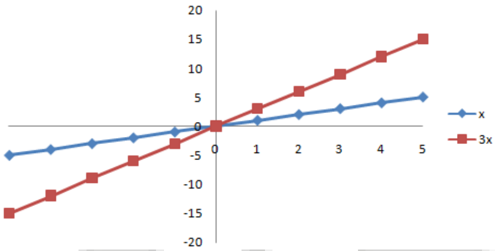
- STEP 1 - reverse x and y:
  $x = 2 + y$
- STEP 2 - solve for y:
  $\mathrm{y} = \mathrm{x} - 2$

The graph of of a function and its inverse functions are mirror images, when mirrored over the line $y = x$. This can also be seen in the graph of the example above.

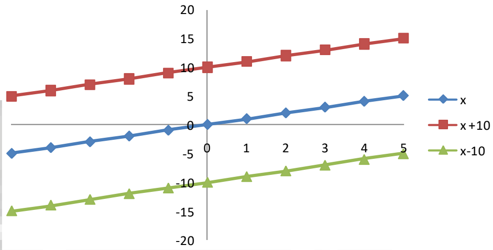
Linear functions

An example of a linear equation is $y = 3x + 2$. This type of equation will be a straight line, when plotted as a graph. The **linear equation** with unknown x is an equation in the form of $f(x) = a \cdot x + b$, where a and b are constants ($a \neq 0$).

For any unit increase in the independent variable x, the output or dependent variable y or $f(x)$ increases by the same **coefficient**/gradient a. Thus, a is the **gradient** (slope) of the linear function. This determines the **slope** (steepness) of this function.



The constant b is the **intercept** of the linear function, i.e. the point at which the linear line intersects/cuts the **y-axis**.



Generally, the following rules apply regarding the intercept (b):

- Increasing the value of b shifts the function parallel up
- Decreasing the value of b shifts the function parallel down

In order to solve for which number the linear function, e.g. $y = 3x + 2$, interscepts/cuts the **x-axis**, the function (y) can be set equal to 0. For this example the calculation would be:

- $y = 3x + 2 = 0$
- $3x + 2 = 0$
- $3x = -2$
- $x = -\frac{2}{3}$

Systems of linear equations

A **system of linear equations** occurs when several linear functions, in the same axis system, need to be satisfied simultaneously. These systems can be solves both algebraically and graphically.
There are only three options possible when solving a system of two linear equations with two unknowns. Those possible options are the following:

- No solution:
  The two graphs never intersect, they are parallel to each other; i.e. they have the same gradient (a) but different intercepts (b).
- One solution:
  The two graphs intersect and their intersection (x, y) is the solution to the system of equations; i.e. the two graphs have different gradients (a).
- Infinitely many solutions:
  The two equations actually describe one function and hence any point (x, y) satisfies both of the equations; i.e. the two graphs have the same gradient (a) and intercept (b).

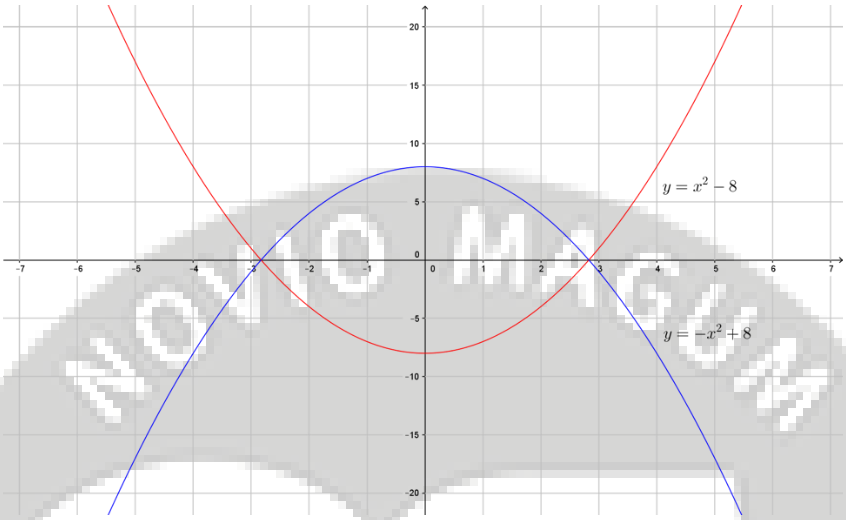
For example, the following system of linear equations can be solves algebraically:

- Take the following system of linear equations:
  $\begin{cases} 2x + 4y = 20 \\ 3x - 2y = 14 \end{cases}$
- STEP 1 - isolate one of the variables (x or y), in this case in the first equation:
  $2x + 4y = 20$
  $x = 10 - 2y$
- Which gives the following system of equations:
  $\begin{cases} x = 10 - 2y \\ 3x - 2y = 14 \end{cases}$
- STEP 2 - substitute $x = 10 - 2y$ into the second equation:
  $3(10 - 2y) - 2y = 14$
  $30 - 8y = 14$
  $-8y = -16$
  $y = 2$
- Knowing that $y = 2$ and $x = 10 - 2y$, the variable x can be calculated:
  $x = 10 - 2y$
  $x = 10 - 2 \cdot (2)$
  $x = 10 - 4$
  $x = 6$
- Therefore, it can be concluded that in this system of linear equations
  $x = 6$ and $y = 2$

Quadratic functions

Next to linear functions, there are other types of functions, such as the **quadratic function**. A quadratic function f is given by $f(x) = a \cdot x^2 + b \cdot x + c$, with a, b and c being constant, real numbers and $a \neq 0$.

The graph associated with such a function is called a **parabola**. When $a > 0$ it is called a **parabola opening upwards** (red graph) and when $a < 0$ it is called a **parabola opening downwards** (blue graph).



To find out what the zero point of a quadratic function are, we need to find the intersection of the parabola with the x-axis (or $f(x)$ equal to 0), just like with a linear function. Here are a two of ways to do this:

- The ABC-formula
- Factorization

The ABC-formula
When solving the quadratic equation $f(x) = a \cdot x^2 + b \cdot x + c = 0$, the well-known ABC-formula can be used:

$$x = \frac{-b \pm \sqrt{D}}{2 \cdot a}$$

In this formula D stands for $D = b^2 - 4 \cdot a \cdot c$, so that gives:

$$x = \frac{-b \pm \sqrt{b^2 - 4 \cdot a \cdot c}}{2 \cdot a}$$

In this formula the $\pm$ indicates that the formula needs to be solved twice, one with a $+$ and once with a –, instead of the $\pm$.

The D indicates the possible amount of solutions when solving a quadratic equation:

- <u>No solution:</u>
  $D < 0$
- <u>One solution:</u>
  $D = 0$, then the only solution can be found in the form of $-\frac{b}{2 \cdot a}$

- Two solutions:
  $D > 0$

Factorization

Next to the ABC-formula, quadratic functions can be solved using factorization. The factorization can be done in two ways:

- Taking the shared factor out of the brackets
  In the equation $f(x) = x^2 + 3x = 0$, the shared factor is x. This factor can be taken out of the brackets, which will result in: $x(x + 3) = 0$
  It can then be concluded that $x = 0$ and $x + 3 = 0$, i.e. $x = -3$
- The sum-product method
  The equation $a \cdot x^2 + b \cdot x + c = 0$, with $a = 1$, can be resolved to $(x + y) \cdot (x + z) = 0$. Where $y + z = b$ and $y \cdot z = c$. For example, $x^2 + 8x + 12 = 0$, becomes $(x + 6) \cdot (x + 2) = 0$. It can then be concluded that $x + 6 = 0$, i.e. $x = -6$, and $x + 2 = 0$, i.e. $x = -2$

Exponential and logarithmic functions

In addition to the linear and quadratic functions, there are exponential and logarithmic functions. The form of an **exponential function** is $f(x) = a^x$, with the variable (x) in the exponent and the constant a as the base of the function.

Logarithmic functions are the inverse functions of exponentials, and are of the form $f(x) = \log_g(x)$. This particular logarithmic function, $f(x) = \log_g(x)$, is the inverse of function $f(x) = g^x$.

The power-rules

When calculating with powers, i.e. exponents, the following rules apply:

- $(x^a)^b = x^{a \cdot b}$ → $x^{a^b} = x^{a \cdot b}$
- $(x \cdot y)^a = x^a \cdot y^a$
- $x^a \cdot x^b = x^{a+b}$
- $\frac{x^a}{x^b} = x^{a-b}$
- $x^0 = 1$, with $x \neq 0$

The logarithm-rules

When calculating with logarithms the following rules apply:

- $\log_a(x \cdot y) = \log_a(x) + \log_a(y)$
- $\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$
- $\log_a(x^n) = n \cdot \log_a(x)$

Euler's number

If M dollars is invested into a bank account, which pays an annual interest rate r that is n-times compounded over the year, then at the end of the year you will have a savings account of:

$$M \cdot \left(1 + \frac{r}{n}\right)^n$$

It is a mathematical fact that as n becomes bigger and bigger, $\left(1 + \frac{r}{n}\right)^n$ gets closer and closer to e^r. So, continuously compounded interest results in $M \cdot e^r$.

Lecture 2

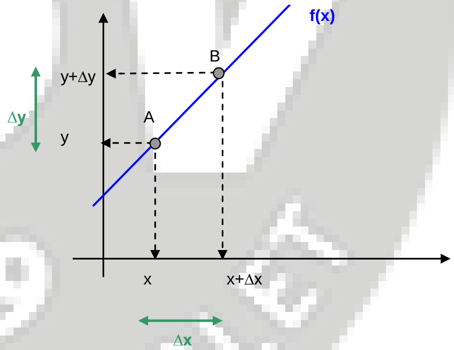
Derivatives.

Introduction to derivatives

Economists often ask how one variable, i.e. the independent variable, affect the value of another variable, i.e. the dependent variable. In order to find the answer to this question, the gradient (a) can be used, which is introduced in Lecture 1 – linear functions.

For example, find the gradient of the linear function $f(x) = a \cdot x + b$ at point A and point B.

- Point A:
  $y = a \cdot x + b$
- Point B:
  $y + \Delta y = a \cdot (x + \Delta x) + b$

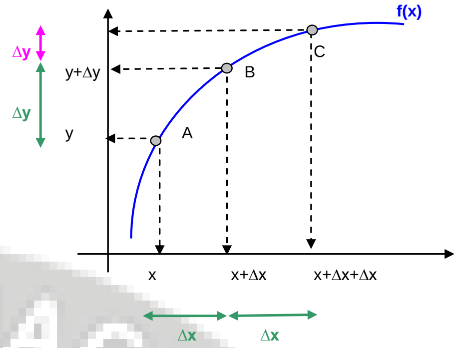


Using the two equations of the example, the gradient can be expressed as $a = \frac{\Delta y}{\Delta x}$

The impact of a change in the independent variable on the dependent variable for a linear function is thus given by the gradient.

Non-linear functions

However, this rule does not hold for non-linear functions. Then, the same increase in the independent variable (Δx) does not always result in the same increase of the dependent variable (Δy). This can be seen in the graph.



Therefore, it needs to be specified how a non-linear function responds to a change in the independent variable (x) at every point of the function $f(x)$. Defining the new function $f'(x)$ can describe this impact of a change relationship. This new function, $f'(x)$, is called the **derivative** of a function.

When differentiating a function, the **first order derivative** will be calculated.

Notation of derivatives
Various notations can be used for a derivative of $f(x)$, but all of them mean the same:

- $f'(x) = \cdots$
- $f_x = \cdots$
- $\frac{df(x)}{dx} = \cdots$
- $\frac{d}{dx}f(x) = \cdots$
- $\frac{dy}{dx} = \cdots$

Useful to know
Economists often use "**marginal**" in front of a name of function. This, marginal function, is the function that is derived from the original function by taking a derivative.

Differentiation rules

There are a number of rules that are needed to be able to differentiate functions:

Main rule of differentiation
$$\frac{d}{dx}x^a = a \cdot x^{a-1}$$

For example, $h(x) = -7x^4$ gives $h'(x) = -28x^3$

Sum rule
The derivative $(f(x) + g(x))'$ of the sum of two functions, $f(x) + g(x)$ is given by
$$\big(f(x) + g(x)\big)' = f(x)' + g(x)'$$

For example, $h(x) = -3x^2 + 2x$ gives $h'(x) = -6x + 2$

Product rule
The derivative $\big(f(x) \cdot g(x)\big)'$ of the product of two functions, $f(x) \cdot g(x)$, is given by
$$\big(f(x) \cdot g(x)\big)' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

For example, $h(x) = (x^2 + 2)(2x - 4)$ gives $h'(x) = 2x \cdot (2x - 4) + 2 \cdot (x^2 + 2)$

Quotient rule

The derivative of $\frac{f(x)}{g(x)}$ is found by using the quotient rule:

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{\left(g(x)\right)^2}$$

For example, $h(x) = \frac{2x-5}{x-5}$ gives $h'(x) = \frac{2\cdot(x-5)-1\cdot(2x-5)}{(x-5)^2}$

Chain rule

Functions can be composed of several different elementary functions, which we can differentiate. We use the chain rule to take a derivate of such functions. The chain rule let us splits up our functions. Take $f(x) = f(u(x))$. Then we have to use the chain rule to find derivative of f with respect to x:

$$\frac{df(x)}{dx} = \frac{df(u)}{du} \cdot \frac{du}{dx}$$

For example, $h(x) = (4x-3)^2$ gives $h'(x) = 2 \cdot (4x-3)^{2-1} \cdot 4$

Exponential and logarithmic differentiation rules

- $h(x) = g^x$ gives the derivative $\frac{dh(x)}{x} = g^x \cdot \ln(g)$
- $h(x) = e^x$ gives the derivative $\frac{dh(x)}{x} = e^x$
- $h(x) = \ln(x)$ gives the derivative $\frac{dh(x)}{x} = \frac{1}{x}$
- $h(x) = \log_g(x)$ gives the derivative $\frac{dh(x)}{x} = \frac{1}{x \cdot \ln(g)}$

Useful to know

- $\frac{1}{x} = x^{-1}$
- $\sqrt{x} = x^{\frac{1}{2}}$

Higher order derivatives

Taking the derivative of the first order derivative will give the **second order derivative**. Differentiating a derivative will thus give a higher order derivative.

For example, find the second order derivative of the function $f(x) = 4\mathrm{x}^3 - 6\mathrm{x}^2 + 2$

- STEP 1 - the first order derivative:
  $f'(x) = 12x^2 - 12x$
- STEP 2 - the second order derivative:
  $f''(x) = 24x - 12$

This process of differentiating can also be expressed as follows:

- $f'(x) = \frac{dy}{dx}$
- $f''(x) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$

Lecture 3

Partial derivatives with one variable.

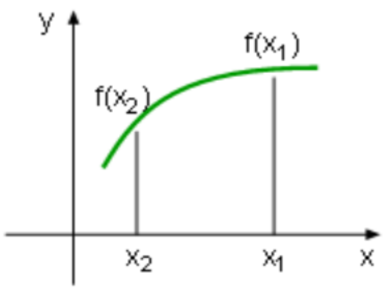
In the previous section, Lecture 2, the derivative was introduced. Derivatives can be used to find function properties, such as the monotonicity, the curvature, minima and maxima. This will be discussed in the following section.

Monotonicity and the first order derivative of a function

As mentioned in “function properties”, Lecture 1, monotonicity explains in which direction a function changes, i.e. if it increases or decreases. The monotonicity of a function can be determined using the first order derivative:

<u>Determining if a function is increasing:</u>

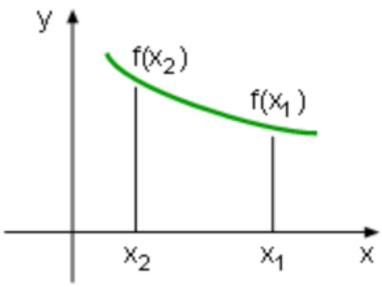
- Take a point on the function $f(x)$.
- Draw a tangent at the point:
    - The slope of the tangent is positive:
      The gradient of the tangent $\frac{\Delta y}{\Delta x}$ is positive (for any $\Delta x > 0$, also $\Delta y > 0$).
      → The derivative of $f(x)$ is positive.



Increasing Function

<u>Determining if a function is decreasing:</u>

- Take a point on the function $f(x)$.
- Draw a tangent at the point:
    - The slope of the tangent is negative:
      The gradient of the tangent $\frac{\Delta y}{\Delta x}$ is negative (for any $\Delta x > 0$, also $\Delta y > 0$).
      → The derivative of $f(x)$ is negative.



Decreasing Function

Monotonicity, using first order derivatives

- If $f'(x) > 0$ on interval (a, b), then $f(x)$ is strictly **increasing** in (a, b).
- If $f'(x) < 0$ on interval (a, b), then $f(x)$ is strictly **decreasing** on (a, b).
- If $f'(x) = 0$ at point x, then we speak of a **stationary point**.

Curvature and the second order derivative of a function

As mentioned in "function properties", Lecture 1, curvature explains at what speed a function increases or decreases. The curvature of a function can be determined using the second order derivative. The **second order derivative** ($f''(x)$), is the derivative of the first order derivative ($f'(x)$). It can be said that $f''(x)$ is the gradient of $f'(x)$.

The function properties of convex and concave are used to indicate the curvature of the function. For a recap, the definitions of these properties can be found in the section Lecture 1 – function properties.

- If $f''(x) > 0$ on interval (a, b), then $f(x)$ is strictly **convex** in (a, b).
- If $f''(x) < 0$ on interval (a, b), then $f(x)$ is strictly **concave** on (a, b).
- If $f''(x) = 0$ at point x, then we speak of a **point of inflection**.

Optimization of a function of one variable

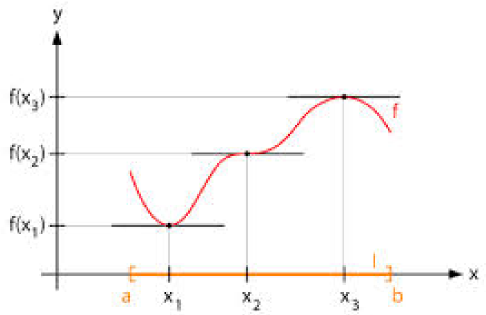
In economics, it is often asked for what values of the independent variable (x) the function obtains the most extreme value(s), i.e. **minima** and **maxima**. Combining the previous two sections, about monotonicity and curvature, makes it possible to find those extreme values. The process of finding the maximum and/or minimum value is called **optimization**.

The maxima or minima of the function $f(x)$ can be found by equalizing the derivative ($f'(x)$) to 0. The derivative indicates the slope of the function at a certain point. In a maximum or minimum, the slope is equal to 0, i.e. a stationary point.

For example, determine the possible maxima and/or minima of the function
$f(x) = 4x^3 - 6x^2 + 2x$

- STEP 1 - find the first order derivative
  $f'(x) = 12x^2 - 12x$
- STEP 2 - equalize the first order derivative to 0
  $f'(x) = 12x^2 - 12x = 0$
  $12x^2 - 12x = 0$
  $12x \cdot (x - 1) = 0$
  $12x = 0$ and $x - 1 = 0$
  $x = 0$ and $x = 1$

However, stationary points are not always minima (x_1) or maxima (x_3), they could also be inflection points (x_2). The second order derivative can determine if it is a maximum, minimum or inflection point.



For example, determine the maxima and/or minima and/or inflection points of the function $f(x) = 4x^3 - 6x^2 + 2x$

- STEP 1 - find the second order derivative
  $f''(x) = 24x - 12$
- STEP 2 - fill the points found in the previous example into the second order derivative
  $f''(0) = 24 \cdot 0 - 12 = -12$
  $f''(1) = 24 \cdot 1 - 12 = 12$
- STEP 3 - determine if the points are maxima, minima or inflection points
  $f''(0) = -12$ The second order derivative of $x = 0$ is smaller than 0. This indicates that the function has **the tendency to go down**. Therefore, it can be concluded that this is a maximum.
  $f''(1) = 12$ The second order derivative of $x = 1$ is bigger than 0. This indicates that the function has **the tendency to go up**. Therefore, it can be concluded that this is a minimum.

If the second order derivative is equal to zero, i.e. $f''(x) = 0$, then the stationary point is neither a maximum nor a minimum, but an inflection point.

Application: elasticity

Economists are often interested in the slope, i.e. the monotonicity of functions. However, in economics the impact of one variable, e.g. price, on a function, e.g. demand/supply, is quantified by **elasticity** rather than the slope.

Why elasticity?
The slope of a function changes when the independent variable is rescaled, e.g. the prices are recalculated from euros to dollars. However, elasticity does not change when such a rescaling procedure is applied.

Elasticity equation

Elasticity (ε) is given by $\frac{\frac{dQ}{dp}}{\frac{Q}{p}}$, which can be rewritten as:

$$\varepsilon = \frac{dQ}{dp} \cdot \frac{p}{Q}$$

The elasticity of a function could be thought of as a <u>percentage change</u> in the dependent variable that results from a <u>percentage change</u> in the independent variable.

The price elasticity (ε) describes how demand reacts to price changes. Demand can be **unit-elastic**, **inelastic** and **elastic**:

- <u>Unit-elastic:</u> $|\varepsilon| = 1$
  One unit of price change induces one unit change of quantity demanded.
- <u>Inelastic:</u> $|\varepsilon| < 1$
  One unit of price change induces less than one unit change of quantity demanded. This are goods that cannot be replaced easily, e.g. medicine and gasoline.
- <u>Elastic:</u> $|\varepsilon| > 1$

One unit of price changes induces more than one unit change of quantity demanded. This are goods that can be substituted with other goods.

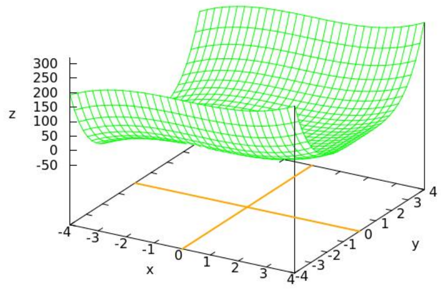
Lecture 4

Functions of more than one variable.

Functions of two and more variables

So far, only functions with one variable have been discussed, but in economics there are also functions with two variables. The production function $f(L, K)$, with output f as a dependent variable and the two independent variables L (labour) and K (capital), is an example of such a function.

These functions have thus more variables than the functions previously discussed. Therefore, it is not possible to visualize the function in a x- and y-axis system. The aforementioned production function ($f(L, K)$) would be plotted in a 3D diagram. Most of the time, one of the variables (e.g. f, L or K) is kept constant. This makes it possible to visualize the function in a x- and y-axis system. These types of curves are called **iso-curves**.



Homogenous functions

Let's take another look at the aforementioned production function, $f(K, L)$. This function is a **Cob-Douglas production function**, which is any function of the form $f(x, y) = A \cdot x^a \cdot y^b$, with the parameters A, a and $b > 0$. In economics it can be interesting to know what happens with the value of a function (f) when all independent variables (x, y or K, L) are multiplied by the same positive number. For some functions this can be calculated. These functions are called **homogenous**.

The definition of **homogeneity** is the following: function $f(x_1, x_2, \dots, x_n)$ is homogeneous of degree t if $f(\lambda x_1, \lambda x_2, \dots, \lambda x_n) = \lambda^t \cdot f(x_1, x_2, \dots, x_n)$ for $\lambda > 0$.

For example, find the homogeneity of the function $f(K, L) = K^{\frac{1}{5}} L^{\frac{2}{5}}$

- $f(t \cdot K, t \cdot L) = 6 \cdot (t \cdot K)^{\frac{1}{5}} \cdot (t \cdot L)^{\frac{2}{5}}$
- $f(t \cdot K, t \cdot L) = 6 \cdot t^{\frac{1}{5}} \cdot K^{\frac{1}{5}} \cdot t^{\frac{2}{5}} \cdot L^{\frac{2}{5}}$
- $f(t \cdot K, t \cdot L) = 6 \cdot t^{\frac{1}{5}+\frac{2}{5}} \cdot K^{\frac{1}{5}} \cdot L^{\frac{2}{5}}$
- $f(t \cdot K, t \cdot L) = 6 \cdot t^{\frac{3}{5}} \cdot K^{\frac{1}{5}} \cdot L^{\frac{2}{5}}$
- $f(t \cdot k, t \cdot L) = t^{\frac{3}{5}} \cdot f(K, L)$
- Therefore, the homogeneity is $\frac{3}{5}$

For homogenous functions, the following applies:

- There are **constant returns to scale** if it is homogeneous of degree $t = 1$
- There are **decreasing returns to scale** if it is homogeneous of degree $t < 1$
- There are **increasing returns to scale** if it is homogeneous of degree $t > 1$

Partial derivatives

Partial derivatives can be used to determine the impact of one independent variable on the value of the function, keeping other things equal.

The partial derivative is the gradient of an iso-curve depicting the relationship between the dependent variable and one independent variable (keeping all other independent variables fixed at some constant level).

The notation of a partial derivative of function f with respect to x is denoted by:

$$\frac{\partial f}{\partial x}$$

The “curly” sign for the derivative means that take all the values of the other remaining variables of function f (other than x) are taken as fixed at some point – i.e. the calculating is done on some iso-curve.

For example, find the partial derivative of function $f(x, y) = 4x^2y^3 + 2x$ with respect to x and with respect to y

- The partial derivative with respect to x is:
  $f_x = 8xy^3 + 2$
- The partial derivative with respect to y is:
  $f_y = 12x^2y^2$

In the same way as a higher order derivative of a function of one variable can be taken, the higher order partial derivative of a function of several variables can be taken. The only thing that has to be kept in mind at every step, is which variable is the relevant variable (of which the derivative is taken). All remaining variables are then taken as given constants.

- The second partial derivative of x with respect to x is:
  $f_{xx} = 8y^3$
- The second partial derivative of y with respect to y is:
  $f_{yy} = 24x^2y$
- The second partial derivative of x with respect to y is:
  $f_{xy} = 24xy^2$

Total derivative

Total derivatives can be used to determine the change of the total value of the function, if all the independent variables are changed.

The total differential captures the changes of a function value if each independent variable changes somewhat. It packages all the partial derivatives of a function together.

Consider function $z = f(x, y)$. Total differential dz is the amount by which a value of function z will change, when the values of all independent variables change by a very small amount.

$dz = \frac{\partial f(x,y)}{\partial x} dx + \frac{\partial f(x,y)}{\partial y} dy$ t the point $(x, y) = (x_0, y_0)$

Implicit derivative

Implicit derivatives can be used to determine the change in the input L needed to keep the value of the function the same, if we change the input K.

The implicit derivative is a derivative that allows the calculating of the slope of an iso-curve (level curve) obtained from a function of several variables.

The slope of an iso-curve if $f(x, y) =$ constant is the following:

- $df = f_x dx + f_y dy$
- $0 = f_x dx + f_y dy$
- $-f_y dy = f_x dx$
- $\frac{dy}{dx} = \frac{-f_x}{f_y}$

Therefore, the implicit derivative of function $f(x, y)$ can be denoted as $\frac{-f_x}{f_y}$

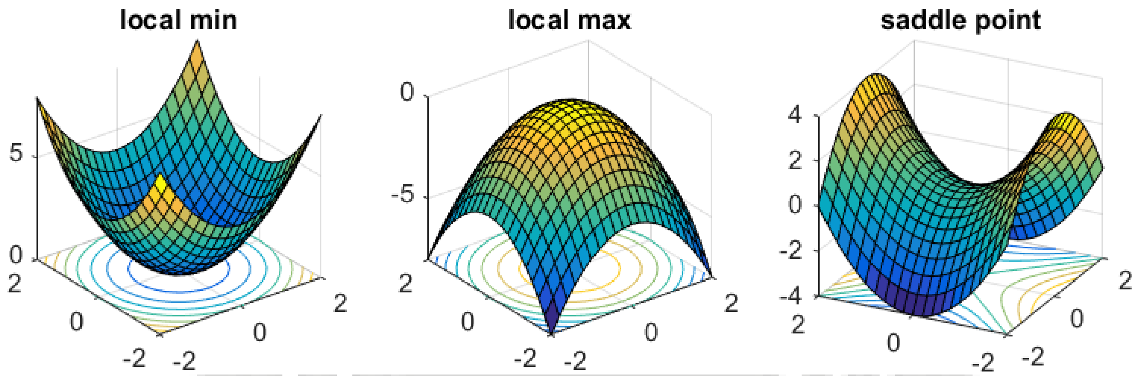
Lecture 5

Optimization of functions of two or more variables.

Introduction to optimization of functions of two or more variables

Lecture 3 discussed finding maxima, minima and inflection points, i.e. optimization, of functions with one variable. The process of optimization can also been done for functions with two or more variables.

These functions can have **local minima**, **local maxima** and **saddle points**, which are illustrated in the graphs below. In addition to the local maxima and minima, there are also global maximum and global minimum. The first, **global maximum**, is the largest of all local maxima of a function. The latter, **global minimum**, is the smallest of all local minima of a function.



When looking at a function with one variable, the derivative of the (entire) function was used to find the maxima and minima (and inflection points). However, there is not one derivative for a function with multiple variables. In order to find the stationary points, partial derivatives are used.

First order condition (FOC)

The first step of optimizing functions with two or more variables, is finding the stationary points. A stationary point can be found by setting the partial derivatives equal to 0.

For example, find the stationary point(s) of the function $f(x,y) = -2x^2 - 2xy - 2y^2 + 36x + 42y - 158$

- STEP 1 - determine the first partial derivative and set it equal to 0
  $f_x = -4x - 2y + 36$
  $f_x = -4x - 2y + 36 = 0$
- STEP 2 - determine the second partial derivative and set it equal to 0
  $f_y = -2x - 4y + 42$
  $f_y = -2x - 4y + 42 = 0$
- STEP 3 – identify the stationary point(s)
  $f_x = -4x - 2y + 36 = 0$
  $2y = -4x + 36$
  $y = -2x + 18$
  $f_y = -2x - 4y + 42 = 0$
  $f_y = -2x - 4 \cdot (-2x + 18) + 42 = 0$
  $-2x + 8x - 72 + 42 = 0$
  $6x = 30$
  $x = 5 \rightarrow y = -2 \cdot 5 + 18 = 8$
  Therefore, the stationary point is at $(5,8)$

Summarily, **the first order condition** can be identified as:

$$\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial y} = 0$$

Second order condition (SOC)

If a function satisfies the first order condition, the stationary points have been found. However, this will not say that all stationary points are minima or maxima. The points could also be saddle points.

In order to determine the nature of the stationary points, **the second order condition** is used. This condition can be written in two different ways. The first is using the **Hessian**, which is a function in the form of $H(x, y) = f_{xx} \cdot f_{yy} - \left(f_{xy}\right)^2$.

- f_{xx} is the derivative of f_x with respect to x
- f_{yy} is the derivative of f_y with respect to y
- f_{xy} is the derivative of f_x with respect to y

Classification of stationary points can be done by inserting the coordinates of the stationary points into the Hessian function. The outcome(s) then determine the nature of the stationary points:

- A positive outcome of the Hessian, and a positive outcome of the function f_{xx}, indicates a minimum.
- A positive outcome of the Hessian, and a negative outcome of the function f_{xx}, indicates a maximum.
- A negative outcome of the Hessian indicates a saddle point.

However, **the second order condition** can also be identified as:

local minimum	**local maximum**
$\frac{\partial^2 z}{\partial x^2} > 0$	$\frac{\partial^2 z}{\partial x^2} < 0$
$\frac{\partial^2 z}{\partial y^2} > 0$	$\frac{\partial^2 z}{\partial y^2} < 0$

… and for both minima and maxima it needs to satisfy:

$$\frac{\partial^2 z}{\partial x^2} \cdot \frac{\partial^2 z}{\partial y^2} > \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2$$

Lecture 6

Constrained optimization.

This lecture on **constrained optimization**, will discuss the determination of maxima and minima of a function that is under a constraint. An example of such a constraint is a maximum amount of money that can be spend.
There are two ways to optimize a function with a constraint:

- The substitution method
- The Lagrange method

Both will be explained, using an example.

Optimization with constraints: substitution method

The **substitution method** will be explained using the production function $f(K, L) = 10K + 10L$ under the constraint $10 \cdot K^{0.5} \cdot L^{0.5} = 800$. Finding the minimum is the aim of this example.

Taking the partial derivatives is not an option for this function. Therefore, the function needs to become univariate, i.e. with one variable. The constraint will be used to do this.

- STEP 1 - express K in L, using the constraint
  $10 \cdot K^{0.5} \cdot L^{0.5} = 800$
  $K^{0.5} \cdot L^{0.5} = 80$
  $K \cdot L = 80^2$
  $K = \dfrac{80^2}{L} = 80^2 \cdot L^{-1}$

- STEP 2 - substitute $80^2 \cdot L^{-1}$ into the production function $f(K, L) = 10K + 10L$
  $f(L) = 10 \cdot 80^2 \cdot L^{-1} + 10L$

This is the univariate function. The next step is differentiating this function, in order to find the minimum.

- STEP 3 - find the derivative of $f(L) = 10 \cdot 80^2 \cdot L^{-1} + 10L$
  $f'(L) = 10 \cdot 80^2 \cdot (-L^{-2}) + 10$

- STEP 4 - set the derivative equal to 0 and solve the equation
  $f'(L) = 10 \cdot 80^2 \cdot (-L^{-2}) + 10 = 0$
  $10 \cdot 80^2 \cdot (-L^{-2}) + 10 = 0$
  $10 \cdot 80^2 \cdot (-L^{-2}) = -10$
  $-\dfrac{10 \cdot 80^2}{L^2} = -10$
  $\dfrac{1 \cdot 80^2}{L^2} = 1$
  $L = 80$

Therefore, the minimum of the function, which satisfies the constrained, is at $L = 80$. The next step is to find K.

- STEP 5 - find K, by using the constraint
  $10 \cdot K^{0.5} \cdot L^{0.5} = 800$
  $10 \cdot K^{0.5} \cdot 80^{0.5} = 800$
  $K^{0.5} \cdot 80^{0.5} = 80$
  $K = 80$

It can be concluded that the minimum at $K = 80, L = 80$, satisfies the constraint.

Optimization with constraints: Lagrange method

The production function in the previous substitution example was not too difficult to optimize since both powers were equal to 0.5. Unfortunately, this will not always be the case. Optimizing more complicated functions can be done using the **Lagrange method**. The Lagrange method will be explained using the production function $f(K, L) = K^{\frac{1}{4}}L^{\frac{3}{4}}$. under the constrained $g(K, L) = K + L = 12$

In order to solve this problem and find the maximum, the Lagrange function will be used. This function is in the form of:

$L(x, y, \lambda) = f(x, y) - \lambda \cdot (g(x, y) - c)$ or $L(x, y, \lambda) = f(x, y) - \lambda \cdot (c - g(x, y))$

In this Lagrange function, the function $f(x, y)$ needs to be optimized and is under the constraint $g(x, y) = c$

Let's solve the example problem:

- STEP 1 - write the Lagrange function
  $L(K, L, \lambda) = K^{\frac{1}{4}} \cdot L^{\frac{3}{4}} - \lambda \cdot (K + L - 12)$

- STEP 2 - set the first order conditions
  $L_K = \frac{1}{4} \cdot K^{-\frac{3}{4}} \cdot L^{\frac{3}{4}} - \lambda \cdot (1) = 0$
  $L_L = \frac{3}{4} \cdot K^{\frac{1}{4}} \cdot L^{-\frac{1}{4}} - \lambda \cdot (1) = 0$
  $L_\lambda = -K - L + 12 = 0$

- STEP 3 - rewrite $L_K = 0$ and $L_L = 0$ to λ
  $L_K = \frac{1}{4} \cdot K^{-\frac{3}{4}} \cdot L^{\frac{3}{4}} - \lambda \cdot (1) = 0 \rightarrow \lambda = \frac{1}{4} \cdot K^{-\frac{3}{4}} \cdot L^{\frac{3}{4}}$
  $L_L = \frac{3}{4} \cdot K^{\frac{1}{4}} \cdot L^{-\frac{1}{4}} - \lambda \cdot (1) = 0 \rightarrow \lambda = \frac{3}{4} \cdot K^{\frac{1}{4}} \cdot L^{-\frac{1}{4}}$

- STEP 4 - set them equal to each other and solve for one of the variables

$$\frac{1}{4} \cdot K^{-\frac{3}{4}} \cdot L^{\frac{3}{4}} = \frac{3}{4} \cdot K^{\frac{1}{4}} \cdot L^{-\frac{1}{4}}$$
$$K^{-\frac{3}{4}} \cdot L^{\frac{3}{4}} = 3 \cdot K^{\frac{1}{4}} \cdot L^{-\frac{1}{4}}$$
$$L^{\frac{1}{4}} \cdot L^{\frac{3}{4}} = 3 \cdot K^{\frac{1}{4}} \cdot K^{\frac{3}{4}}$$
$$L = 3K$$

- STEP 5 - fill this into L_λ and find K, L and λ
  $L_\lambda = -K - L + 12 = 0$
  $-K - 3K + 12 = 0$
  $-4K + 12 = 0$
  $4K = 12$
  $K = 3$
  $L = 3K = 3 \cdot 3 = 9$
  $\lambda = \frac{1}{4} \cdot \mathrm{K}^{-\frac{3}{4}} \cdot \mathrm{L}^{\frac{3}{4}} = \frac{1}{4} \cdot (3)^{-\frac{3}{4}} \cdot (9)^{\frac{3}{4}} \approx 0.57$

It can be concluded that the maximum at $K = 3, L = 9, \lambda = 0.57$, satisfies the constraint.

Summarily, the steps needed to optimize a function under constrained, using the Lagrange method, are the following:

- STEP 1 - write the Lagrange function
- STEP 2 - set the FOCs:

  $L_x(x, y, \lambda) = 0$

  $L_y(x, y, \lambda) = 0$

  $L_\lambda(x, y, \lambda) = 0$
- STEP 3 - rewrite $L_x = 0$ and $L_y = 0$ to λ and set them equal to one another
- STEP 4 - fill this into L_λ and find x, y and λ

Lecture 7

Integrals.

Indefinite integrals

The previous lectures have extensively discussed the concept of differentiation. In economics, it can be also useful to do the opposite of differentiating a function. This process is called **integration**.
Integration is finding the function F that is the **integral** of the function f. The derivative of function F is equal to the original function f, because differentiation and integration are opposites. An integral is sometimes referred to as a **primitive**.

The definition of an integral can be written as follows:
$\int f(x)dx = F(x) + C$ whenever $F'(x) - f(x)$ where C is any constant

The dx that that stands in the integral, indicates that integration is done with respect to the variable x. In addition, behind the integrals stands C, which represents the constant.

There are several functions F of which the derivative is equal to function f. In all these different functions F, the constant C differs. For example, the number 2 could be the constant C. When differentiated, this constant will disappear. Therefore, this constant can be different values for which $F' = f$.
Since there are multiple possible functions F, the constant C is used to indicate them all. Apart from the constant (C), all parts of F are unique so that $F' = f$. Such a function F, with a constant C behind it, is called an indefinite **integral**.

Integration rules

General rules:

- $\int af(x)dx = a\int f(x)dx$ … where a is a constant
- $\int[f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$

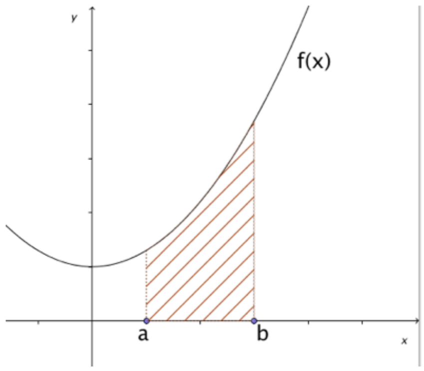
Specific rules:

- $\int x^a dx = \frac{1}{(a+1)}x^{a+1} + C$ … if $a \neq -1$
- $\int \frac{1}{x}dx = ln|x| + C$
- $\int e^{ax}dx = \frac{1}{a}e^{ax} + C$ … if $a \neq -1$
- $\int a^x dx = \frac{1}{ln(a)}a^x + C$ … if $a > 0$ and $a \neq 1$

Definite integrals

In addition to the indefinite integrals, there are definite integrals. The definition of a definite integral is the following:

$$\int_a^b f(x)dx = A$$



Function $f(x)$ is illustrated in the graph. The shaded area under the function $f(x)$, from $x = a$ to $x = b$, can be calculated by finding the primitive function $F(x)$ of function $f(x)$. This particular integral is indicated by $\int_a^b f(x)dx$. Calculating the shaded area is done by: $\int_a^b f(x)dx = F(b) - F(a)$.

For example, calculate the integral $\int_4^8 \frac{1}{x^3}dx$

- $f(x) = \frac{1}{x^3}$ gives $F(x) = -\frac{1}{2x^2}$

- $\int_a^b f(x)dx = \left(-\frac{1}{2\cdot(8)^2}\right) - \left(-\frac{1}{2\cdot(4)^2}\right) = \frac{3}{128}$

Definite integration rules

In short, if $\int f(x)dx = F(x) + C$ then $\int_a^b f(x)dx = F(b) - F(a) \equiv \int_a^b f(x) \equiv [F(x)]_a^b$.
The rules of definite integrals do still apply, but there is no need to include C.

Integration by substitution

Some complicated integrals can be simplified by using the method of integration by **substitution**. The following knowledge can be used:

$\int f(g(x))g'(x)dx = F(g(x)) + C$, where $u = g(x)$ and where $du = g'(x)$

For example, solve $\int_1^5 \frac{4\,dx}{(4x+5)^2}$ using substitution.

- $4x + 5 = u(x)$
- $4dx = du$
- $dx = \frac{du}{4}$
- $\int_1^5 \frac{4}{(u)^2} \cdot \frac{du}{4} = \int_1^5 u^{-2}du = [-u^{-1}]_1^5$
- $[-u^{-1}]_1^5 = [-(4x + 5)^{-1}]_1^5$
- $[-(4x + 5)^{-1}]_1^5 = -\left(\frac{1}{25} - \frac{1}{9}\right) = \frac{16}{225}$

Application: consumer and producer surplus

Let's look at how the consumer and producer surplus can be calculated using integration.

$D(q)$, i.e. the demand function, tells the maximum price consumers would be willing to pay for each unit - even if they now only have to pay p^*; because of that, the difference between $D(q)$ and p^* is called the **consumer surplus**.
Economists are interested in calculating this difference between $D(q)$ and p^* (i.e. consumer surplus). Formally, this area is the integral of $D(q)$ between 0 and q^* minus the rectangle p^* times q^*.

$CS = \int_0^{q^*} D(q)dq - p^*q^*$

Similarly, $S(q)$, i.e. the supply function, tells the minimum price producers have to be paid for each unit in order to produce it - even if they now receive p^* for every produced unit; because of that, the difference between p^* and $S(q)$ is called the **producer surplus**.

Economists are interested in calculating this difference between $S(q)$ and p^* (i.e. producer surplus). Formally, this area is the integral of p^* times q^* minus the integral of $S(q)$ between 0 and q^*.

$$PS = p^*q^* - \int_0^{q^*} S(q)dq$$

Lecture 8

Matrices – part 1.

Introduction to matrices

Matrices are collections of numbers. For example, a matrix with the name A:

$$A = \begin{pmatrix} 4 & 0 \\ 3 & 1 \\ -1 & 2 \end{pmatrix}$$

Matrix A has the **dimension ($\boldsymbol{n \times m}$)** if it has n rows and m columns. In this case the dimension of the matrix A is (3x2).
The **element $\boldsymbol{a_{ij}}$** of the matrix can be found on the i-th row and the j-th column. For example, in matrix A the element a_{12} is 0.

In mathematics, matrices can be used for all sorts of things. For instance, matrices are a useful notation for systems of equations, which will be examined further in this lecture and Lecture 9. In order to use matrices to solve systems of equations, a couple of basic matrix operations should be explained.

The sum of two matrices
It is possible to calculate the sum of two matrices <u>of the same size</u>. The corresponding elements can then be added up, i.e. a_{12} of the first matrix plus a_{12} of the second matrix.

$$\begin{pmatrix} 1 & -1 & 6 \\ 2 & 0 & 0 \\ 4 & 0 & -5 \end{pmatrix} + \begin{pmatrix} 1 & 5 \\ -2 & 0 \\ -1 & 0 \end{pmatrix} = \text{NO ANSWER}$$

$$\begin{pmatrix} 4 & 0 \\ 3 & 1 \\ -1 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 5 \\ -2 & 0 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 5 & 5 \\ 1 & 1 \\ -2 & 2 \end{pmatrix}$$

The multiplication of a matrix by a number
It is possible to multiply a matrix by a number. All the elements of the matrix are then multiplied by the number.

$$3 \cdot \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 10 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 0 & 0 \\ 30 & 6 \end{pmatrix}$$

The transposing of a matrix

It is possible to transpose a matrix. The **transpose** of matrix A is denoted by A' or A^T. Matrix A' is transpose of A if $a'_{ij} = a_{ji}$. In addition, if matrix A is of the dimension ($n \times m$), then matrix A' is of the dimension ($m \times n$).

$$A = \begin{pmatrix} 2 & -3 & -6 \\ 1 & 0 & 7 \end{pmatrix} \text{ gives } A' = \begin{pmatrix} 2 & 1 \\ -3 & 0 \\ -6 & 7 \end{pmatrix}$$

The multiplication of two matrices

It is possible to multiple two matrices. This matrix operation is slightly more complicated than the previous operations. The multiplication of two matrices will be discussed and explained using an example.

$$A = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 3 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 3 & 1 \\ 2 & 1 \\ 1 & 0 \end{pmatrix}$$

$$A \cdot B = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} 3 & 1 \\ 2 & 1 \\ 1 & 0 \end{pmatrix}$$

The first step is to determine whether the multiplication is possible.
Take the matrix A with the dimension (**2** × 3) multiplied by the matrix B with the dimension (2 × **2**). <u>The multiplication is allowed when matrix A has as many rows as matrix B has columns, which is the case for A times B.</u> However, it is not possible to multiply matrix B (**2** × 2) by matrix A (2 × **3**). Therefore, the order of the multiplication matters.

The second step is to determine the dimension of the final matrix (C).
When matrix A is multiplied by matrix B, the final matrix C will have the dimension of ($rows_A \times columns_B$). In this case, the final matrix C will have the dimension of (2 × 2).

The third step is to calculate the elements of the final matrix.
The element c_{ij} of the new matrix is constructed by multiplying the i-th row of matrix A with the j-th column of matrix B.For example, calculate element c_{11}. Take the element of the first row of A and the element of the first column of B, i.e. $\begin{pmatrix} 1 & 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$.

The first elements can be multiplied with each other, i.e. $1 \cdot 3$, to which the product of the second elements can be added, which will be followed by the third elements. This gives $c_{11} = 1 \cdot 3 + 0 \cdot 2 + 2 \cdot 1 = 5$. When this process is repeated for each row/column, the result will be the following matrix C:

$$C = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} 3 & 1 \\ 2 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1\cdot 3 + 0 \cdot 2 + 2 \cdot 1 & 1 \cdot 1 + 0 \cdot 1 + 2 \cdot 0 \\ -1 \cdot 3 + 3 \cdot 2 + 1 \cdot 1 & -1 \cdot 1 + 3 \cdot 1 + 1 \cdot 0 \end{pmatrix} = \begin{pmatrix} 5 & 1 \\ 4 & 2 \end{pmatrix}$$

Type of matrices

There are multiple types of matrices:

- Square matrix → a matrix with the same number of rows and columns, thus dimension ($n \times n$)
- Identity matrix → a square matrix, with all zero's and only one's on the main diagonal
- Diagonal matrix → a matrix with any number on the diagonal allowed and only zeros on the off-diagonal terms
- Upper triangular matrix → a matrix with only zeros below the diagonal terms, and any number elsewhere
- Symmetrix matrix → a matrix A where $a_{ij} = a_{ji}$

Matrices and systems of equations

Matrices can be used to solve systems of equations. There three important things to know, before using matrices to solve the systems.

Augmented matrix
An **augmented matrix** for the system of equations $A \cdot x = b$ is a matrix of the coefficients A that multiplies the vector of variables x, with an added part behind a vertical line. This added part is the column vector of results.

This explanation may be difficult, but a visual representation of an augmented matrix will make it clearer:

For the system of equations $\begin{cases} -2y + 5z = 11 \\ x + y + 3z = 12 \\ 4x - z = 1 \end{cases}$, the augmented matrix will be:

$$\left(\begin{array}{ccc|c} 0 & -2 & 5 & 11 \\ 1 & 1 & 3 & 12 \\ 4 & 0 & 1 & 1 \end{array}\right)$$

Gaussian elimination method
The **Gaussian elimination method** can be used to solve a system of equations, such as $A \cdot x = b$. This method can be used as follows.

- The first step is to apply the allowed operations to a system of equations in such a way that one variable will be isolated.
- The second step is to use this variable in all remaining equations, and then repeat the approach of step one.

This method thus allows for a step-by-step elimination of the variables in order to find a solution.

Solving a system of equation
When solving a system of n equations and m variables there can be three possible outcomes:

- If $n < m$, there are infinitely many solutions. This is an **underdetermined system**.
- If $n = m$, there is one solution. This is a **determined system**.
- If $n > m$, there is no solution. This is an **overdetermined system**.

Lecture 9

Matrices – part 2.

Determinants

Whether a system of equations, such as $A \cdot x = b$, has one, none or infinitely many solutions is determined by a number of a matrix A which is called the **determinant**. The determinant is thus a number that describes an important property of a square matrix and makes it possible (1) to find whether there is a unique solution to a system of equations (determinant) and (2) to calculate the solution of a system of equations, if there is one (inverse matrix).

The determinant is denoted by $\det(A)$ or $|A|$. When the determinant is equal to zero, there is no solution. This determinant can be calculated in three different ways, which will be discussed below.

1 × 1 matrices
The determinant of a matrix A for $n = 1$ is equal to $\det(A) = a_{11}$. This system of equations has a solution if $a_{11} \neq 0$.

2 × 2 and 3 × 3 matrices
The determinant of a matrix A for $n = 2$, the determinant can be calculated using the following "trick".

The red arrow, going diagonally from left to right, crosses the elements which will be multiplied and then get a positive sign in front of them.
The blue arrow, going diagonally from right to left, crosses the elements which will be multiplied and then get a negative sign in front of them.

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Therefore, the determinant of a matrix A for $n = 2$ is equal to $\det(A) = a_{11}a_{22} - a_{21}a_{12}$.
The system of equations has a solution if $\det(A) \neq 0$.

This "trick" can also be used when calculating the determinant of a matrix A for $n = 3$.

The red arrows, going diagonally from left to right, cross the elements which will be multiplied and then get a positive sign in front of them.
The blue arrows, going diagonally from right to left, cross the elements which will be multiplied and then get a negative sign in front of them.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Therefore, the determinant of a matrix A for $n = 3$ is equal to

$\det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$.
The system of equations has a solution if $\det(A) \neq 0$.

Expansion by co-factors/minors

Another general method to calculate the determinant is **expansion by co-factors**. The expansion by co-factors method will be explained using a matrix A for $n = 3$.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

This matrix can be broken down into smaller matrices, which will make it easier to calculate the determinant.

The first step is to choose one row or column of the matrix. In this example, the first row $(1,2,3)$ will be used. Take the first element out of this row, which is 1, and multiply this with the remaining matrix. This remaining matrix consist of all the elements that are not in the row or column of the selected element.

$$A = \begin{pmatrix} \cancel{1} & \cancel{2} & \cancel{3} \\ \cancel{4} & 5 & 6 \\ \cancel{7} & 8 & 9 \end{pmatrix}$$

Thus, $1 \cdot \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix}$

The straight lines around the matrices indicates that a determinant is calculated.

Repeat this process for the entire row and the determinant will be:

$|A| = 1 \cdot \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \cdot \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \cdot \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} =$

$1 \cdot (5 \cdot 9 - 6 \cdot 8) - 2 \cdot (4 \cdot 9 - 6 \cdot 7) + 3 \cdot (4 \cdot 8 - 5 \cdot 7) = 0$

Determining if the multiplication needs a $+$ or a $-$ in front of it, can be done using the position of the selected element. The first element (1) stands on position a_{11} in matrix A. When the numbers in the subscript are added to one another, the result is 2. **The equal number indicates a $+$**. The second element (2) stands on position a_{12} in matrix A. When the numbers in the subscript are added to one another, the result is 3. **The unequal number indicates a $-$.**

In general, this method can be used for any row or column of matrix A.

Basic rules of determinants

When the determinant needs to be calculated for large matrices, e.g. $n > 3$, it can be convenient to simplify the matrix. The elementary (matrix) operations can be used to do this.

The impact of elementary operations on the matrix A on the value of $\det(A)$ are:

- Adding a multiple of one row to another row → will result in no change of the determinant

- Interchanging two rows of a matrix → multiply the determinant by (-1)
- Multiply row by constant a → multiply the determinant by constant a

Moreover, if the elements of an entire row or column of a matrix A are 0, then $\det(A) = 0$. Next to that $\det(A)$ is equal to $\det(A')$ and $\det(A) \cdot \det(B) = \det(A \cdot B)$.

Inverse matrix

The knowledge gained about matrices and determinants can now be applied. Take the following system of equations:

$$\begin{cases} 4x - 4y = -1 \\ \;\; 2x + y = 4 \end{cases}$$

This system of equations can be written as the matrix $A = \begin{pmatrix} 4 & -4 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$ in which the first matrix is called A, the second x and the third b. This results in the equation $A \cdot x = b$.

To solve the system of equations, x needs to be found. The x can be expressed in A and b:

- $A \cdot x = b$
- $x = \frac{b}{A}$
- $x = A^{-1} \cdot b$

Therefore, finding A^{-1} and multiplying this with b will give the solution of the system of equations. This A^{-1} is called the **inverse matrix** of matrix A.

2 × 2 matrices

In order to calculate the inverse matrix of A for $n = 2$, e.g. for the example above, the determinant is needed first.

$A = \begin{pmatrix} 4 & -4 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$, gives the determinant $|A| = a_{11}a_{22} - a_{12}a_{21} = 4 \cdot 1 - (-4 \cdot 2) = 12$. The determinant is bigger than zero, therefore the system of equations has a solution and the calculation of the inverse can be continued.

The solution matrix of A for $n = 2$ is equal to $\frac{1}{|A|} \cdot \begin{pmatrix} \boldsymbol{a_{22}} & \boldsymbol{-a_{12}} \\ \boldsymbol{-a_{21}} & \boldsymbol{a_{11}} \end{pmatrix}$, which is in this case $\frac{1}{12} \cdot \begin{pmatrix} 1 & 4 \\ -2 & 4 \end{pmatrix} = \frac{\begin{pmatrix} 1 & 4 \\ -2 & 4 \end{pmatrix}}{12}$. This results in $A^{-1} = \begin{pmatrix} \frac{1}{12} & \frac{1}{3} \\ -\frac{1}{6} & \frac{1}{3} \end{pmatrix}$.

The solution of the system of equations can then be determined by solving $x = A^{-1} \cdot b$, which gives $\begin{pmatrix} \frac{1}{12} & \frac{1}{3} \\ -\frac{1}{6} & \frac{1}{3} \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 4 \end{pmatrix} = \begin{pmatrix} \frac{1}{12} \cdot -1 + \frac{1}{3} \cdot 4 \\ -\frac{1}{6} \cdot -1 + \frac{1}{3} \cdot 4 \end{pmatrix} = \begin{pmatrix} \frac{5}{4} \\ \frac{3}{2} \end{pmatrix}$.

Therefore, it can be concluded that $x = \frac{5}{4}$ and $y = \frac{3}{2}$.

3 × 3 matrices

Finding the inverse matrix to matrix A for $n = 3$ is slightly more complicated than for 2 × 2 matrices. The first step is to calculate the determinant, in order to know if there is a solution to the system of equations. If $|A| \neq 0$, the inverse matrix can be calculated by using multiple elementary operations on the matrix A, and simultaneously to the identity matrix. The goal is to transform the matrix A into the identity matrix. From this transformation, the inverse matrix will be received.

For example, find the inverse of matrix $A = \begin{pmatrix} 1 & 3 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{pmatrix}$

- STEP 1 - write the identity matrix next to matrix A
  $$\left(\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 3 & 4 & 0 & 1 & 0 \\ 1 & 4 & 3 & 0 & 0 & 1 \end{array}\right)$$
- STEP 2 - subtract row one from row two
  $$\left(\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 1 & 4 & 3 & 0 & 0 & 1 \end{array}\right)$$
- STEP 3 - subtract row one from row three
  $$\left(\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right)$$
- STEP 4 - subtract three times row three from row one
  $$\left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 4 & 0 & -3 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right)$$
- STEP 5 - subtract three times row two from row one
  $$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 7 & -3 & -3 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right)$$
- STEP 6 - switch row two and three
  $$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 7 & -3 & -3 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right)$$

The inverse A^{-1} is $\begin{pmatrix} 7 & -3 & -3 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$

Solving systems of equations: Cramer's rule

Systems of equations can also be solved without using inverse matrices. A method that is often used, is **Cramer's rule**.

2 × 2 matrices
This method will be explained using the same system of equations as in the beginning of this lecture.

The system of equations can be written as the matrix $A = \begin{pmatrix} 4 & -4 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$ in which the first matrix is called A, the second x and the third b. This results in the equation $A \cdot x = b$. With $\det(A) = 12$. The system of equations is solves when the values of x are found.

According to Cramer's rule if matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$, then $D_x = det \begin{bmatrix} p & b \\ q & d \end{bmatrix}$ and $D_y = det \begin{bmatrix} a & p \\ c & q \end{bmatrix}$. So, $x = \frac{D_x}{|A|} = \frac{pd-bq}{ad-bc}$ and $y = \frac{D_y}{|A|} = \frac{aq-pc}{ad-bc}$

For the example this would mean, $x = \frac{-1\cdot 1-(-4\cdot 4)}{12} = \frac{5}{4}$ and $y = \frac{4\cdot 4-(-1\cdot 2)}{12} = \frac{3}{2}$

3 × 3 matrices
The Cramer's rule can also be expressed more generally: the matrix D_j is obtained from matrix A by replacing its j-th column (of A) by the column vector b. The solution to the system $A \cdot x = b$ is then given by $x_j = \frac{D_j}{|A|}$ for each of the variables.

For example, solve the following system of equations for y using Cramer's rule

$$\begin{cases} 2x + 4y + 15z = -1 \\ -8x - 14y - 65z = 0 \\ -6x - 70z = -20 \end{cases}$$

- STEP 1 - rewrite the system of equations to matrices
$$A = \begin{pmatrix} 2 & 4 & 15 \\ -8 & -14 & -65 \\ -6 & 0 & -70 \end{pmatrix}, x = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, b = \begin{pmatrix} -1 \\ 0 \\ -20 \end{pmatrix}$$

- STEP 2 - calculate the determinant of A, using the method of expansion by co-factors
$$|A| = 2 \cdot -14 \cdot -70 + 4 \cdot -65 \cdot -6 + 15 \cdot -8 \cdot 0 - (15 \cdot -14 \cdot -6) - (2 \cdot -65 \cdot 0) - (4 \cdot -8 \cdot -70) = 20$$

- STEP 3 - to calculate the variable y, the y column in matrix A will be replaced with matrix b

replacing $\begin{pmatrix} 4 \\ -14 \\ 0 \end{pmatrix}$ with $\begin{pmatrix} -1 \\ 0 \\ -20 \end{pmatrix}$ gives $D_y = det\begin{pmatrix} 2 & -1 & 15 \\ -8 & 0 & -65 \\ -6 & -20 & -70 \end{pmatrix}$

- STEP 4 - calculate the determinant of D_y, using the method of expansion by co-factors
  $$|D_y| = 2 \cdot 0 \cdot -70 + -1 \cdot -65 \cdot -6 + 15 \cdot -8 \cdot -20 - (15 \cdot 0 \cdot -6) - (2 \cdot -65 \cdot -20) - (-1 \cdot -8 \cdot -70) = -30$$
- STEP 5 - determine y
  $y = \frac{D_y}{|A|} = -\frac{30}{20} = -\frac{3}{2}$

Mathematics vocabulary

Mathematics vocabulary used in the lectures/summary

$\forall$	for all
$\exists$	there exists
Σ	sum
Π	product
$\emptyset$	empty set
$\mathbb{R}$	set of all real numbers
$\mathrm{s.t.}$	such that
$\in$	belongs to
$\notin$	does not belong to
Δ	delta (change)
$\Leftrightarrow$	whenever (if and only if, under no other conditions)
$\Rightarrow$	then (implies that)